TOPOLOGY - III, EXERCISE SHEET 1

Exercise 1. *Quotient topology* (easy)

Definition 1.1: Let X be a topological space and let $\sim$ be an equivalence relation on X. Let q be the function $q : X \to X/\sim$ which sends an element $x \in X$ to it's equivalence class $[x] \in X/\sim$. Recall that one can naturally endow the set $X/\sim$ with a topology as follows. We declare a subset U to be open in $X/\sim$ if and only if $q^{-1}(U)$ is open in X.

(1) Show that the quotient of a connected (respectively compact) space is connected (respectively compact).
(2) The quotient of a Hausdorff space need not be Hausdorff. Let $X := \mathbb{R} \times \{0, 1\}$. We declare $(r, 0) \sim (s, 1) \iff r = s$ and $r \neq 0$. The quotient $Y := X/\sim$ is commonly referred to as the line with two origins. Prove that Y is not Hausdorff.
(3) Give an example of a space X and an equivalence relation $\sim$ such that the projection map $q : X \to X/\sim$ is not an open map.
(4) If a group G acts on a topological space X, one can endow the space of orbits of the action with the quotient topology. We denote this space by X/G. We define the n-Torus $\mathbb{T}_n$ to be the $n-$ fold product $S^1 \times ... \times S^1$. Recall the universal property of the quotient of a topological space and use it to define a homeomorphism $\mathbb{R}^n/\mathbb{Z}^n \xrightarrow{\sim} \mathbb{T}_n$.
(5) If Y is a subset of a topological space X then often by the quotient X/Y we mean the quotient $X/\sim$ where $x_1 \sim x_2 \iff x_1 \in Y$ and $x_2 \in Y$. Intuitively this is the space we obtain from gluing all the points of Y to a point. Show that the quotient $(\mathbb{R} \times S^1)/(\{0\} \times S^1)$ is homeomorphic to the double cone in $\mathbb{R}^3$ cut out by the equation $Z^2 - X^2 - Y^2 = 0$.

Exercise 2. *Covering Spaces and the Homotopy lifting Property* (medium)

Definition 2.0: Let $p : E \to X$ be a function. We say that a function $\overline{f} : Y \to E$ is a lift of a function $f : Y \to X$ if $f = p \circ \overline{f}$.

Definition 2.1: We say a continuous map $p : E \to X$ is a covering space map if p is surjective and for every $x \in X$ there exists an open neighbourhood U of x such that $p^{-1}(U) = \sqcup_i V_i$ is a disjoint union of open sets of E and p induces a homeomorphism $V_i \xrightarrow{\sim} U$.

Prove the **homotopy lifting property** for covering spaces:
Given a covering space $p : E \to X$ and a continuous map $F : Y \times [0, 1] \to X$ such that $F(y, 0) : Y \to X$ lifts to a continuous map $\overline{F}_0 : Y \to E$. Show that F has a unique lift $\overline{F} : Y \times [0, 1] \to E$ such that $\overline{F}(y, 0) = \overline{F}_0$.

Hint: Do this for a trivial covering space, where E is just the discrete disjoint union of some copies of X. Then for the general case try to break up the interval $[0,1]$ into compact subintervals and try to replicate the same proof locally.

Exercise 3. *Consequences of the Homotopy Lifting Property* (medium)
Let $p : E \to X$ be a covering space map.

(1) Let $a : [0,1] \to X$ be a path with starting point $a_0 := a(0)$ and fix $e_0 \in p^{-1}(a_0)$. Show that a lifts to a unique path $\overline{a} : [0,1] \to E$ such that $\overline{a}(0) = e_0$.
(2) Let $x_0 \in X_0$ and let $e_0 \in p^{-1}(x_0)$. If two paths $a, b : [0,1] \to X$ with the same starting point x_0 are homotopy equivalent, then show that their unique lifts with starting point e_0 are also homotopy equivalent.
(3) Prove that the quotient map $p : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is a covering map. Using this and part (1) and (2), compute the fundamental group of $\mathbb{T}_2$.
(4) Use part (2) to conclude that the induced map $p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ is injective.
(5) Construct an action of the group $\pi_1(X, x_0)$ on the set $p^{-1}(x_0)$ using the homotopy lifting property. This action is called the monodromy action.
  (a) Prove that the kernel of this action is $p_*(\pi_1(E, e_0))$.
  (b) Show that the monodromy action is transitive if the covering space E is path connected.
  (c) Let X be a connected space, then show that the function $x \mapsto |p^{-1}(x)|$ is constant. The value of this function is called the number of sheets of the covering $p : E \to X$. Suppose E is path connected, use parts (a) and (b) to show that the number of sheets is equal to the index of $p_*(\pi_1(E, e_0))$ in $\pi(X, x_0)$.

Important Remark on the general theory of covering spaces:
Part (4) of Exercise 3 gives a way to associate a subgroup of $\pi_1(X, x_0)$ to a covering space of X. If we impose some mild conditions on X, namely we ask X to be path connected and locally simply connected (recall that simply connected means that the fundamental group is trivial) then we can associate a covering space to a subgroup of $\pi_1(X, x_0)$:

By assuming the above conditions on X we can construct a simply-connected cover $\overline{X}$ of X called the universal cover. A subgroup $H \subseteq \pi_1(X, x_0)$ acts on $\overline{X}$ by restricting the monodromy action to H. Then X/H turns out to be a path connected covering space of X with fundamental group isomorphic to H. In this way there exists a one to one correspondence:

$$\{\text{Isomorphism classes of connected covers of X}\} \leftrightarrow \{\text{Conjugacy classes of Subgroups of } \pi_1(X, x_0)\}.$$

This is a bit involved to give as an exercise however the interested reader can find details in the section titled “Classification of covering spaces” in Hatcher’s book starting from page 63.